2021

MATHEMATICS — HONOURS

Third Paper

(Module – V)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group – A

[Modern Algebra – II]

(Marks : 15)

Answer ***any three*** questions.

1. (a) Let $(G, *)$ be a finite cyclic group of order n. Then prove that for every positive divisor d of n there exists a unique subgroup of G of order d.

(b) Prove or disprove : If G is a commutative group of order 6 and has an element of order 3, then G is cyclic. 3+2

2. (a) Prove that the order of each subgroup of a finite group is a divisor of the order of the group.

(b) Find the images of the elements 3 and 4 if $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & & & 3 \end{pmatrix}$ is an odd permutation. 3+2

3. (a) Show that the ring of matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ contains no divisor of zero if $a, b \in \mathbb{Q}$ but contains divisor of zero if $a, b \in \mathbb{R}$.

(b) Show that the field of rational numbers has no proper sub-field. 3+2

4. (a) State Lagrange's theorem and establish that the converse of Lagrange's theorem is not true.

(b) In the symmetric group S_5, solve the equation $x(1\ 2\ 3) = (2\ 4\ 3)$. 3+2

5. Prove that a finite integral domain is a field. 5

Group – B

[Linear Programming and Game Theory]

(Marks : 35)

Answer ***any five*** questions.

6. (a) A manufacturer produces two types of commodities X and Y. Production cost of one unit of commodities X and Y are Rs. 1,000 and Rs. 1,500, respectively, and times needed are 6 hours and 8 hours, respectively. He can work 8 hours per day and his capital is Rs. 20,000. The profit on one unit of X and Y are Rs. 100 and Rs. 150, respectively. The problem is to determine the number of units of X and Y to be produced by the manufacturer per week in order maximize his profit per week. Formulate the problem as an L.P.P.

(b) Solve the following L.P.P. graphically :

Maximize $z = 2x_1 + 4x_2$

subject to $x_1 + 2x_2 \leq 5,$

$x_1 + x_2 \leq 4,$

$x_1, x_2 \geq 0.$ 4+3

7. (a) Define a convex set and an extreme point of a convex set. Give example of a convex set which has no extreme point.

(b) Show that $x_1 = 2, x_2 = 1, x_3 = 3$ is a feasible solution of the system of equations

$$4x_1 + 2x_2 - 3x_3 = 1$$
$$6x_1 + 4x_2 - 5x_3 = 1$$

Reduce it to a basic feasible solution of the system. [(1+1)+1]+4

8. Find the optimal solution of the following L.P.P. by solving its dual : 7

Maximize $z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \leq 10,$

$2x_1 + 3x_2 \leq 18,$

$x_1 \leq 8,$

$x_2 \leq 6,$

$x_1, x_2 \geq 0.$

9. Solve the following L.P.P. by Big-M method : 7

Maximize $z = 2x_1 - 3x_2$

subject to $-x_1 + x_2 \geq -2,$

$5x_1 + 4x_2 \leq 46,$

$7x_1 + 2x_2 \geq 32,$

$x_1, x_2 \geq 0.$

10. Find the optimal solution of the following transportation problem and find the Minimum cost of transportation : 7

					Supply (a_i)
	2	2	2	1	3
	10	8	5	4	7
	7	6	6	8	5
Demand (b_j)	4	3	4	4	

11. Solve the following travelling salesman problem : 7

	A	B	C	D	E
A	∞	2	4	7	1
B	5	∞	2	8	2
C	7	6	∞	4	6
D	10	3	5	∞	4
E	1	2	2	8	∞

12. Consider the L.P.P. : Maximize $z = c^T x$ subject to $Ax = b$, $x \geq 0$. If, for any basic feasible solution x_B of the L.P.P., $z_j - c_j \geq 0$ for every column a_j of A, then prove that x_B is an optimal solution. [Symbols have their usual meanings] 7

13. (a) If $\left(a_{ij}\right)_{m\times n}$ be the pay-off matrix of a two-person zero sum game, prove that

$$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}.$$

(b) In a rectangular game, the pay-off matrix is given by

$$\begin{bmatrix} 10 & 5 & 5 & 20 & 4 \\ 11 & 15 & 10 & 17 & 25 \\ 7 & 12 & 8 & 9 & 8 \\ 5 & 13 & 9 & 10 & 5 \end{bmatrix}$$

State, giving reasons, whether the players will use pure or mixed strategies. What is the value of the game? 3+4

14. (a) Prove that, if we add a fixed number P to each element of a pay-off matrix then the optimal strategies remain unchanged while the value of the game is increased by P.

(b) Using mixed strategies, find the optimal strategies and the value of the game for the following game, whose pay-off matrix is given by $\begin{bmatrix} 6 & -4 \\ -1 & 2 \end{bmatrix}$. 3+4